An inverse Hodge problem and some solutions from hypergeometric motives

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Varieties and their Hodge numbers

Let $\mathcal{X}$ be a smooth projective variety over $\mathbb{C}$. Then its cohomology groups have a Hodge decomposition:

$$H^w(\mathcal{X}, \mathbb{C}) = \bigoplus_{p=0}^{w} H^{p,w-p}.$$

A typical first step in understanding a variety $\mathcal{X}$ is to understand its Hodge numbers $h^{p,q} := \dim_{\mathbb{C}}(H^{p,q})$. It is convenient to think of these as forming a Hodge diamond. If $\dim_{\mathbb{C}}(\mathcal{X}) = 3$, this diamond is:

$$\begin{array}{ccccccc}
 & & & h^{3,3} & & & \\
 & & h^{2,3} & & h^{3,2} & & \\
 & h^{1,3} & & h^{2,2} & & h^{3,1} & \\
h^{0,3} & & h^{1,2} & & h^{2,1} & & h^{3,0} \\
 & h^{0,2} & & h^{1,1} & & h^{2,0} & \\
 & & h^{0,1} & & h^{0,1} & & \\
 & & & h^{0,0} & & &
\end{array}$$

Familiar varieties and their Hodge diamonds

Genus g curve: $\begin{array}{ccc} & 1 & \\ g & & g \\ & 1 & \end{array}.$

K3 surface: $\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}.$

Quintic hypersurface: $\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 1 & & 0 & \\ 1 & & 101 & & 101 & & 1 \\ & 0 & & 1 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}.$

Abelian fourfold: $\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 4 & & 4 & & & \\ & & 6 & & 16 & & 6 & & \\ & 4 & & 24 & & 24 & & 4 & \\ 1 & & 16 & & 36 & & 16 & & 1 \\ & 4 & & 24 & & 24 & & 4 & \\ & & 6 & & 16 & & 6 & & \\ & & & 4 & & 4 & & & \\ & & & & 1 & & & & \end{array}.$

In the sequel, we consider only varieties X defined over $\mathbb{Q}$ and write $\mathcal{X} = X(\mathbb{C})$.

Motivic decomposition

Having in mind connections with automorphic representations and L-functions, we henceforth work in the category $\mathcal{M}(\mathbb{Q},\mathbb{Q})$ of motives defined over $\mathbb{Q}$ with coefficients in $\mathbb{Q}$. For X as before, its cohomology $H^w(\mathcal{X},\mathbb{Q})$ is an object in $\mathcal{M}(\mathbb{Q},\mathbb{Q})$ having a Hodge vector

$$(h^{w,0},\ h^{w-1,1},\ \ldots, h^{1,w-1},\ h^{0,w}).$$

But $H^w(\mathcal{X},\mathbb{Q})$ may decompose in $\mathcal{M}(\mathbb{Q},\mathbb{Q})$, giving a decomposition of its Hodge vector.

Example A: X is a genus g curve with Jacobian factoring into abelian varieties of dimension g_1 and g_2:

$$H^1(\mathcal{X},\mathbb{Q}) = M_1 \oplus M_2, \qquad (g,g) = (g_1,g_1) + (g_2,g_2).$$

Example B: X is a K3 surface with Néron-Severi rank ρ:

$$H^2(\mathcal{X},\mathbb{Q}) = M_{\mathrm{trans}} \oplus M_{\mathrm{alg}}, \qquad (1,20,1) = (1,20-\rho,1) + (0,\rho,0).$$

Fullness as a non-degeneracy condition

An irreducible motive $M \in \mathcal{M}(\mathbb{Q}, \mathbb{Q})$ of weight w and rank n has an associated compact group G, its Sato-Tate group. If w is odd, $G \subseteq Sp_n$. If w is even, $G \subseteq O_n$. We say that M is *full* if $G = Sp_n$, SO_n, or O_n.

Revisiting three equations considered before, the motives in red are definitely not full while motives in blue are typically full:

Curve X with factorizing Jacobian:	$H^1(\mathcal{X}, \mathbb{Q}) = M_1 \oplus M_2$,
$K3$ surface X with $\rho \geq 2$:	$H^2(\mathcal{X}, \mathbb{Q}) = M_{\mathrm{trans}} \oplus M_{\mathrm{alg}}$,
Abelian fourfold X:	$H^4(\mathcal{X}, \mathbb{Q}) = \Lambda^4 H^1(\mathcal{X}, \mathbb{Q})$.

In many situations, the natural expectation is that a given motive M is full and proving fullness is often easy.

Motives from modular forms with trivial character

Let w be an odd positive integer. Then isomorphism classes of motives $M \in \mathcal{M}(\mathbb{Q}, \mathbb{Q})$ with Hodge vector

$$(1, \overbrace{0, \ldots, 0}^{w-1}, 1)$$

are in bijection with normalized Hecke newforms f of modular weight $w + 1$ on $\Gamma_0(N)$ with rational coefficients.

Non-full M correspond to forms f with complex multiplication. They exist for all w.

Full M correspond to forms f without complex multiplication. They exist for $w = 1, 3, 5, 7, \ldots, 43, 45, 47, 49$, and I conjecture they do not exist for $w \geq 51$.

The Hodge inverse problem

Consider vectors $h = (h^{w,0},\ h^{w-1,1},\ \dots, h^{1,w-1},\ h^{0,w}) \in \mathbb{Z}_{\geq 0}^{w+1}$ with $w \geq 1$, $h^{p,q} = h^{q,p}$, and (to normalize) $h^{w,0} \geq 1$.

Problem. *Given h, does there exist a full motive M in $\mathcal{M}(\mathbb{Q}, \mathbb{Q})$ with Hodge vector h?*

Instances:

h	Answer
(g, g)	Yes, from curves
$(1, a, 1)$	Yes, from K3 surfaces if $a \leq 19$
$(1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1)$	Yes, from the Ramanujan form
$(1, \text{(fifty 0's)}, 1)$	Likely no.
$(4, 1, 1, 3, 2, 1, 1, 2, 3, 1, 1, 4)$	?? at first, but Yes from HGMs
$(1, 0, 4, 0, 0, 0, 0, 4, 0, 1)$	?? at first, but Yes from HGMs

Two dichotomies

h is called *rigid* if it has an interior zero and *mobile* else. From an algebro-geometric perspective, it should be harder to find motives for rigid h, because Griffiths traversality prevents them from moving in families.

Examples all with rank four:

	mobile	rigid
regular	$(1,1,1,1)$	$(1,0,0,1,0,0,0,0,0,0,1,0,0,1)$
irregular	$(2,2)$	$(2,0,0,0,0,0,0,2)$

h is called *regular* if all its entries are 0's or 1's, except perhaps for a central 2. It is called *irregular* else. It may be harder to use representation theory to find motives in $\mathcal{M}(\mathbb{Q},\mathbb{Q})$ for irregular h, because they can't be isolated by the trace formula.

Formalism of hypergeometric motives (HGMs)

Let n be a positive integer. Let $f(x), g(x) \in \mathbb{Z}[x]$ be coprime monic polynomials of degree n with all roots being roots of unity. Then for any $t \in \mathbb{Q} - \{0, 1\}$ one has a corresponding rank n motive $H(f(x), g(x), t) \in \mathcal{M}(\mathbb{Q}, \mathbb{Q})$.

Example with $n = 6$:

$$\begin{aligned} f(x) &= \Phi_2(x)^2\Phi_8(x) = (x+1)^2(x^4+1), \\ g(x) &= \Phi_3(x)^2\Phi_6(x) = (x^2+x+1)^2(x^2-x+1), \\ t &= 4/3. \end{aligned}$$

Magma allows many computations with HGMs. For example:

```
H := HypergeometricData([2,2,8],[3,3,6]);
L := LSeries(H,4/3);
EulerFactor(L,7);
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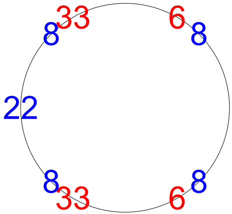
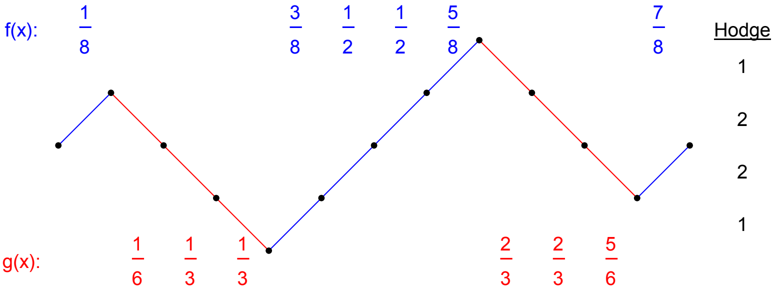
$1 + 12x - 2\cdot 7^2x^2 - 59\cdot 7^2x^3 - 2\cdot 7^5x^4 + 12\cdot 7^6x^5 + 7^9x^6$

Hodge numbers of HGMs

The Hodge vector of an HGM is independent of the specialization point t. It is calculated by how the roots of $f(x)$ and $g(x)$ intertwine on the unit circle, as illustrated by $f(x) = \Phi_2(x)^2\Phi_8(x)$ and $g(x) = \Phi_3(x)^2\Phi_6(x)$:





Mobile Hodge vectors in ranks ≤ 24

Proposition. *In ranks ≤ 24, every mobile Hodge vector h comes from a full hypergeometric motive, except the following twelve Hodge vectors, all orthogonal:*

<u>Rank 20</u>

$(6, 1, 1, 1, 2, 1, 1, 1, 6)$

<u>Rank 22</u>

$(6, 1, 1, 1, 1, 2, 1, 1, 1, 1, 6)$

$(4, 1, 2, 1, 1, 1, 2, 1, 1, 1, 2, 1, 4)$

<u>Rank 23</u>

$(1, 21, 1)$

<u>Rank 24</u>

$(9, 1, 1, 2, 1, 1, 9)$

$(7, 1, 1, 1, 1, 2, 1, 1, 1, 1, 7)$

$(6, 1, 1, 1, 1, 1, 2, 1, 1, 1, 1, 1, 6)$

$(5, 1, 2, 1, 1, 1, 2, 1, 1, 1, 2, 1, 5)$

$(4, 1, 3, 1, 1, 1, 2, 1, 1, 1, 3, 1, 4)$

$(1, 6, 1, 1, 1, 1, 2, 1, 1, 1, 1, 6, 1)$

$(4, 1, 1, 2, 1, 1, 1, 2, 1, 1, 1, 2, 1, 1, 4)$

$(4, 1, 2, 1, 1, 1, 1, 2, 1, 1, 1, 1, 2, 1, 4)$

Ranks ≤ 24, continued

A family $H(f(x), g(x))$ is called *primitive* if $f(x)/g(x)$ is not a function of x^k for some $k \geq 2$. The Zariski closure of the monodromy group of the primitive family $H(f(x), g(x))$ is the entire symplectic or orthogonal group whenever $w \geq 1$. This implies $H(f(x), g(x), t)$ is full for "almost all" specialization points t.

The proposition is then proved by direct computation. For example, there are 319,685,444 symplectic families in rank 24, but only $2^{11} = 2048$ Hodge vectors. The average fiber size is thus

$$r_{24} = \frac{319685444}{2048} \approx 156096.$$

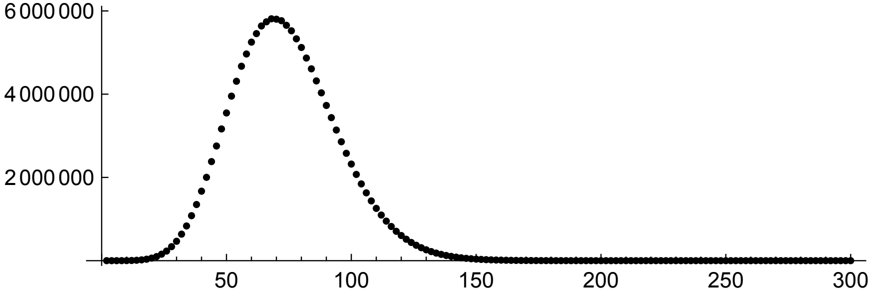
Thus it is not surprising that all fibers are non-empty, as asserted by the proposition. In fact, the smallest fibers occur above

$$(5, 1, 2, 1, 1, 2, 2, 1, 1, 2, 1, 5) \text{ and } (8, 1, 1, 2, 2, 1, 1, 8)$$

and have size 34.

Larger ranks

The average number of symplectic rank n families per Hodge vector is r_n, graphed as follows.



The maximum ratio on the picture is $r_{68} \approx 5,810,819$. The last data point is $r_{300} \approx 0.000013$. So in large ranks, hypergeometric motives answer the Hodge inverse problem positively only for a vanishingly small fraction of mobile Hodge vectors.

Rigid solutions from HGMs at $t = 1$

One can also specialize hypergeometric families at the mild singular point $t = 1$.

All Hodge numbers stay the same except:

- When w is even, the central Hodge number drops by 1, as in $(2, 3, 1, 3, 2) \to (2, 3, 0, 3, 2)$.
- When w is odd, the two centermost Hodge numbers drop by 1, as in $(2, 3, 1, 1, 3, 2) \to (2, 3, 0, 0, 3, 2)$.

Fullness fails in the *reflexive* case $f(x) = (-1)^n g(-x)$, because of an operator inherited from $t \mapsto 1/t$.

Outside the $w = 0$, the imprimitive, and the reflexive cases, it seems likely that fullness holds for all but finitely many $(f(x), g(x))$. Fullness can be verified for given $(f(x), g(x))$ by computing two sufficiently different Euler factors, as on the next slide.

Rigid solutions from HGMs at $t = 1$, continued

Example:

```
H:=HypergeometricData([3,3,3,3],[1,1,1,1,1,1,1,1]);
L:=LSeries(H,1);
```

The Hodge vector is $(1,1,1,0,0,1,1,1)$ by the above procedure.

```
f2 := EulerFactor(L,2); f2;
```

$1 + 9x + 39 \cdot 2x^2 + 207 \cdot 2^3x^3 + 39 \cdot 2^8x^4 + 9 \cdot 2^{14}x^5 + 2^{21}x^6$

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f5 := EulerFactor(L,2); f5;
```

$1 + 18x - 4416 \cdot 5x^2 + 65592 \cdot 5^3x^3 - 4416 \cdot 5^8x^4 + 18 \cdot 5^{14}x^5 + 5^{21}x^6$

Both polynomials are conformally even sextics, so their Galois group is within $\mathrm{Weyl}(Sp_6) = 2^3 : S_3$ of order 48. The biggest their joint Galois group could be is $48 \cdot 48 = 2304$. Indeed:

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Order(GaloisGroup(f2*f5));
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This suffices to show that $H((x^2 + x + 1)^4, (x - 1)^8, 1)$ is full.

Rigid solutions from reflexive HGMs

For n even, reflexive motives $H(f(x), f(-x), 1)$ decompose as the sum of two motives in $\mathcal{M}(\mathbb{Q}, \mathbb{Q})$ of equal or near-equal ranks.

Example. For the rank 14 motive $M = H((x-1)^{16}, (x+1)^{16}, 1)$, the decomposition on Hodge vectors is

$$\begin{aligned}(1,1,1,1,1,1,1,0,0,1,1,1,1,1,1,1) &= \\ (1,0,1,0,1,0,1,0,0,1,0,1,0,1,0,1) &+ \\ (0,1,0,1,0,1,0,0,0,0,1,0,1,0,1,0)&\end{aligned}$$

For $p = 3$, 5, the Euler factor $\det(1 - x\mathrm{Fr}_p|M)$ factors into an irreducible sextic and an irreducible octic. The nature of the irreducible factors confirms both fullness and the Hodge numbers.

Rigid solutions from reflexive HGMs

In general, we conjecture that the decomposition $h = h_1 + h_2$ has the "maximal fairness" property:

The numbers $h_1^{p,q} - h_2^{p,q}$ *are all in* $\{-1, 0, 1\}$ *with the non-zero differences alternating in sign for* $p \geq q$*.*

We also conjecture fullness of each summand outside of a finite number of exceptions.

This would give positive solutions to the Hodge inverse problem for infinitely many difficult-looking h. For example, we'd expect that the two summands of

$$H(\Phi_{27}(x)(x+1)^{16}, \Phi_{54}(x)(x-1)^{16}, 1)$$

are both full, with Hodge vectors

$$(3,2,1,0,0,0,0,1,2,3) \text{ and } (2,3,0,1,0,0,1,0,3,2).$$

Selected References

The talk is presently being converted to a paper.

For motives with Hodge number $(1, 0, ..., 0, 1)$:

Newforms with rational coefficients. To appear in the Ramanujan Journal.

Hodge number formula in:

Roman Fedorov. *Variations of Hodge structures for hypergeometric differential operators and parabolic Higgs bundles.* To appear in International Mathematics Research Notices. Antecedents include works of Terasoma, Corti, Golyshev, Dettweiler, and Sabbah.

The HGM package in *Magma* is by Mark Watkins. The L-function package is by Tim Dokchitser.